Attenuation Constants of Waveguides With General Cross Sections[1]

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Abstract:

A numerical technique for the computation of power transfer and the attenuation constants of waveguides with arbitrary cross section is studied. The wave function, solution of the scalar Helmholtz wave equation, is obtained for the ideal waveguide with perfectly conducting walls by the point-matching method. Using the conventional assumption that the field inside the waveguide with walls of finite but large conductivity is practically the same as that in the ideal waveguide, the power transfer and the attenuation constant are formulated. Numerical values are obtained for the TE_{10} mode of the square waveguide to demonstrate the accuracy of the method.

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1. Introduction

In the investigation of electromagnetic wave propagation in practical waveguides, the ideal guides with perfectly conducting walls of the same geometrical configuration are usually considered first. The field distributions and the cutoff wave numbers are approximately the same for both guides if the practical guide walls are made of good conductors. It is reasonable to approximate the power transfer and the current flow for the practical case by those of the ideal solutions, and therefore, the attenuation constant can be estimated.

When the waveguides under consideration have arbitrary cross sections, non-separability of the wave equations in the cross-sectional coordinate systems is encountered. Approximate techniques for solving such problems have been investigated by several authors [Cohn, 1947; Yashkin, 1958; Meinke et al., 1963; Tischer and Yee, 1964] to overcome this difficulty. Recently, the point-matching method has been applied to calculate the cutoff frequencies of the waveguides [Yee and Audeh, 1965]. It is a technique by which the wave function satisfies the boundary conditions at a finite number of points.

In practical considerations, waveguides are made of good conducting material with finite conductivity. The power dissipated in the guide walls is, therefore, of considerable importance at frequencies in the microwave region or higher. The purpose of this paper is to calculate the power transfer and the attenuation constants of waveguides with arbitrary cross sections by the point-matching method.

A square waveguide is considered as an example to test the theory present here. The field distribution is obtained and compared with the exact solution, and the agreement is excellent. The power transmitted in the guide is also calculated with three-place accuracy. The general attenuation constants for TE and TM modes are formulated and verified by calculating the attenuation constant of a square waveguide operating in the TE_{10} mode and the results agree very well with the ideal solutions.

[redacted]

2. The Point-Matching Solution of the Wave Equation

Consider the air-filled hollow-piped uniform waveguide with a coordinate system as indicated in Fig. 1(a). The cross section of the guide is assumed within the applicability of the point-matching method. Let the electromagnetic wave propagate in the z-direction with a time-harmonic dependence [exp $(j\omega t)$]. As shown by the authors [Yee and Audeh, 1965], the wave function, solution of the scalar Helmholtz wave equation, is conveniently expressed as a series expression in terms of the circular cylindrical modes. If the series converges uniformly and rapidly, the wave function may be approximately written as

$$\psi = \sum_{n=o}^{N} (A_n \cos n\theta + B_n \sin n\theta)\, J_n(kr) \tag{1}$$

where r and θ are the polar coordinates, J_n is the nth order Bessel's function of the first kind, N is an integer depending on the desired accuracy, A_n and B_n are expansion constants to be determined by the boundary conditions. The wave number k, the eigenvalue of the present boundary value problem, is related to the propagation constant k_z as

$$k^2 = k_o^{\,2} - k_z^{\,2}$$

where

$$k_o^{\,2} = \omega^2 \mu_o \epsilon_o$$

$$k_z = 2\pi/\lambda_g$$

μ_o and ϵ_o are the constitutive parameters of air. The quantity λ_g is the guide wavelength. Conventionally, the wave function $\psi = H_z$ for TE (transverse electric) modes, and $\psi = E_z$ for TM (transverse magnetic) modes. The wave function ψ then must satisfy either Dirichlet or Neumann boundary conditions for ideal waveguides.

In applying the point-matching method, the boundary conditions mentioned above are imposed at only (2N + 1) properly chosen points around the cross sectional contour of the waveguide. These boundary conditions are automatically satisfied approximately elsewhere at the surface. Let the points (r_o, θ_o), (r_1, θ_1), $r_2, \theta_2)$, ... (r_{2N}, θ_{2N}) be a set of chosen points around the cross section. A system of (2N + 1) homogeneous linear algebraic equations of the expansion coefficients is obtained from (1) and the proper boundary conditions at these points. That is,

$$\sum_{n=o}^{N} (A_n \cos n\theta_m + B_n \sin n\theta_m) J_n (kr_m) = 0 \tag{2}$$

for TM wave modes, and

$$kr_m \sum_{n=o}^{N} (A_n \cos n\theta_m + B_n \sin n\theta_m) J'_n (kr_m)$$
$$+ \tan a_m \sum_{n=o}^{N} n(-A_n \sin n\theta_m + B_n \cos n\theta_m) J_n(kr_m) = 0 \tag{3}$$

for TE wave modes, where $m = 0, 1, 2 \ldots 2N$, and α_m is the angle between the radial unit vector and the unit vector normal to the cross-sectional contour at point (r_m, θ_m) as shown in Fig. 1(b).

To obtain non-trivial solutions for the expansion coefficients A_n and B_n, the determinant of these coefficients must be zero. The cutoff wave numbers are then readily determined. There are a denumerably infinite number of values of k possible, each of which represents the cutoff wave number of a particular wave mode. Corresponding to each value of k, the expansion coefficients A_n and B_n of (1) can be obtained. This is done by setting one of the coefficients equal to unity and solving the system of 2N inhomogeneous linear equations for the remaining coefficients. The choice of the 2N equations in either (2) or (3) is arbitrary. The wave function is then fully determined for each wave mode. With the knowledge of the wave function the transverse field components can be computed by [Plonsey and Collin, 1961]

$$\bar{E}_t = (jk_z/k^2)\,[\,-\nabla_t E_z + (\omega\mu_o/k_z)\;\bar{z} \times (\nabla_t H_z)] \tag{4}$$

$$\bar{H}_t = (-jk_z/k^2)\,[(\omega\epsilon_o/k_z)\;\bar{z} \times (\nabla_t E_z) + \nabla_t H_z] \tag{5}$$

where $\bar{z}$ is the unit vector in the z-direction, and ∇_t is the transverse gradient operator. The above procedure describes the field inside the waveguide completely.

3. Power Transfer and Attenuation Constant

The field distribution inside practical waveguides made of good conductors can be approximated by that of the corresponding ideal waveguide. Hence, the power transfer is given by

$$P_T = (1/2) \int_S \mathrm{Re}\, [\overline{E}_t \times \overline{H}_t^* \cdot \overline{z}]\, dS \tag{6}$$

where S is the cross-sectional area of the waveguide, (*) denotes the operation of taking the complex conjugate. The transverse components of the field, E_t and H_t are described by (4) and (5) with the point-matching solution of (1). Combining (4) through (6) for each of TE modes and TM modes, yields

$$P_T = (G/k^2) \int_S |\nabla_t \psi|^2\, dS \tag{7}$$

where

$$G = (1/2 Z_o)\, (f/f_c)^2\, \zeta \qquad \text{for TM modes}$$

$$G = (Z_o/2)\, (f/f_c)^2\, \zeta \qquad \text{for TE modes}$$

$$\zeta = \sqrt{1 - (f_c/f)^2}$$

and $Z_o = \sqrt{\mu_o/\epsilon_o}$, is the intrinsic impedance of free space. The quantities f_c and f are the cutoff and operating frequencies respectively. By using Gauss theorem [Ramo and Whinnery, 1958] (7) can be further reduced to

$$P_T = G \int_S |\psi|^2\, dS \tag{8}$$

Evaluation of the integral in (7) or (8) may be accomplished by numerical techniques.

Since the conductivity of the guide walls is finite in practical application, part of the transmitted power will dissipate in the walls. The power loss per unit length of the guide is conventionally estimated by

$$P_L = (R_s/2) \int_C |H_{tan}|^2 \, d\ell \tag{9}$$

where $R_s = \sqrt{\omega\mu_o/2\sigma}$, is the surface resistance of the guide wall, σ is the conductivity of the conducting material. The path, C, of the line integral is the contour of the cross section. The integrand in (9) is the square of the magnitude of the magnetic field component tangential to the periphery of the ideal guide walls. Since the normal component of the transverse magnetic field H_t automatically vanishes at the guide surface, it is then possible to express H_{tan} for TM wave modes as follows:

$$|H_{tan}|^2 = |\overline{H}_t (r_c , \theta)|^2 \tag{10}$$

where r_c, a function of θ, describes the cross-sectional contour. For TE wave modes however, the longitudinal component of the magnetic field also contributes to the tangential component. Hence,

$$|H_{tan}|^2 = |\overline{H}_t (r_c , \theta)|^2 + |\psi(r_c , \theta)|^2 \tag{11}$$

From (5), the square of the magnitude of the transverse magnetic field, therefore, may be written as

$$|\overline{H}_t|^2 = (f/f_c)^2 \, F(r , \theta) \qquad \text{for TM modes} \tag{12}$$

and

$$|\overline{H}_t|^2 = (f\zeta/f_c k^2) \, F(r , \theta) \qquad \text{for TE modes} \tag{13}$$

where $$F(r, \theta) = \left(\frac{\partial \psi}{\partial r}\right)^2 + \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta}\right)^2$$

The attenuation constant is conventionally defined by

$$\alpha = P_L / 2\, P_T \tag{14}$$

if the guide walls are made of good conducting material. Combining (8) through (14) yields the following attenuation constants:

$$\alpha = (R_s / 2Z_o\, \zeta k^2 \int_S |\psi|^2\, dS) \oint_C F(r_c\,,\ \theta)\, r_c\, d\theta \tag{15}$$

for TM wave modes, and

$$\begin{aligned} \alpha = (R_s / 2Z_o\, \zeta \int_S\ |\psi|^2\, dS)[(\zeta/k)^2 \oint_C F(r_c\,,\ \theta)\, r_c\, d\theta \\ + (f_c/f)^2 \oint_C |\psi(r_c\,,\ \theta)|^2\, r_c\, d\theta] \end{aligned} \tag{16}$$

for TE wave modes. The integrations in (15) and (16) can be performed numerically and good accuracy is obtainable as will be demonstrated in the next section.

4. Numerical Example

To demonstrate the validity of the point-matching method for determination of the field distribution, power transfer, and the attenuation constant, it is assumed that an electromagnetic wave is propagating inside a square waveguide in the TE_{10} mode. The guide has a width of 2a and is placed with its center at the origin of a rectangular coordinate system as illustrated in Fig. 2. Since the longitudinal field component H_z is symmetrical with respect to the x-axis for TE_{10}, the sine terms in (1) are omitted. The cutoff wave number calculated by (3) using six points only on the upper half of the guide's cross-sectional contour (see Fig. 2) is 1.5716, compared with the exact value of 1.5708. The expansion coefficients were determined in terms of the coefficient A_1 which is equal to a pre-assigned value of unity. The resulting wave function is therefore expressed in the following form:

$$\psi = H_z = \sum_{n=1}^{3} (-1)^{n+1} J_{2n-1}(kr) \cos(2n-1)\theta \tag{17}$$

with three-place accuracy. The disappearance of the even terms in (17) is not surprising because H_z for TE_{10} is antisymmetric with respect to the y-axis. The series in (17) converges rapidly. It is observed that the accuracy of ψ is estimated by taking the ratio of the lowest neglected term to the first term. That is,

$$J_7(kr) / J_1(kr) < 0.001$$

for the largest value of r which is $\sqrt{2}a$. It can therefore be concluded that the wave function in (17) does represent the field distribution in the square guide under consideration up to three-place accuracy. Table 1 contains the values of H_z as a function of position x for three different locations in the y-direction, and compared with the exact solution

$$H_z = 0.5 \sin(\pi x/2a)$$

The power transported in the waveguide was calculated numerically using (8) and (17), and the result was 0.1256 which is excellent when compared with the exact value of 0.1250.

The attenuation constant is obtained by substituting the approximate wave function (17) into (16) and performing numerical integrations. It is

$$(\alpha_p a Z_o / R_{sc}) = (0.994/2\zeta) \sqrt{f/f_c} \, [1 + 2.014 \, (f_c/f)^2] \tag{18}$$

where $R_{sc} = R_s \sqrt{f_c/f}$, the surface resistance at cutoff frequency, and α_p denotes the attenuation constant of the point matching solution. The exact attenuation constant, α_e, for a square guide of width $2a$ [Ramo and Whinney, 1944] is given by

$$(\alpha_e \, a \, Z_o / R_{sc}) = (1/2\zeta) \sqrt{f/f_c} \, [1 + 2 \, (f_c/f)^2] \tag{19}$$

The comparison of these two attenuation constants is shown in Fig. 3 over the frequency range of $(f/f_c) = 1$ through 10. The validity of the attenuation constant calculated by the point matching solutions is then verified.

5. Discussion

The point-matching method offers a new and convenient method for solving waveguide problems especially when the cross-sectional shapes of such guides are not largely deviated from a circle. The theory has been verified by applying it to well known problems, and it does give reasonably accurate results if a digital computer is available. In Table 1 for example, the point-matching method yields the field distribution in the square guide with excellent agreement with the exact solution at $y/a = 0$, and $y/a = 1$. There is however, a slight deviation near the middle region ($y/a = 0.5$). From a mathematical point of view, the agreement between the field distribution calculated by the approximate and exact approaches is not surprising, since, by wave transformation [Harrington, 1961]

$$\sin x = 2 \sum_{n=1}^{\infty} (-1)^{n+1} J_{2n-1}(x) \cos(2n-1)\theta$$

The formulation of the attenuation constants (15) and (16) is proven correct as seen by the results in Fig. 3, in addition to the agreement of the expressions for TE_{10} of (18) and (19). It may also be noted that the formulas (15) and (16) agree with those formulated by Gannet and Szekely [1960].

Since the integrals in (15) and (16) are frequency-independent, the attenuation constants in the high frequency range increase with frequency for all TM and most of TE modes. The attenuation constant in the TE case is inversely proportional to the frequency if and only if

$$\oint_C F(r_c, \theta)\, r_c \, d\theta = 0 \tag{19}$$

However, this condition in general cannot be fulfilled except for TE_{no} modes of circular guides. Therefore, it is predicted that a wide band of frequency of low attenuation can be achieved if the cross-sectional shape of a uniform waveguide is not largely deviated from a circle.

6. References

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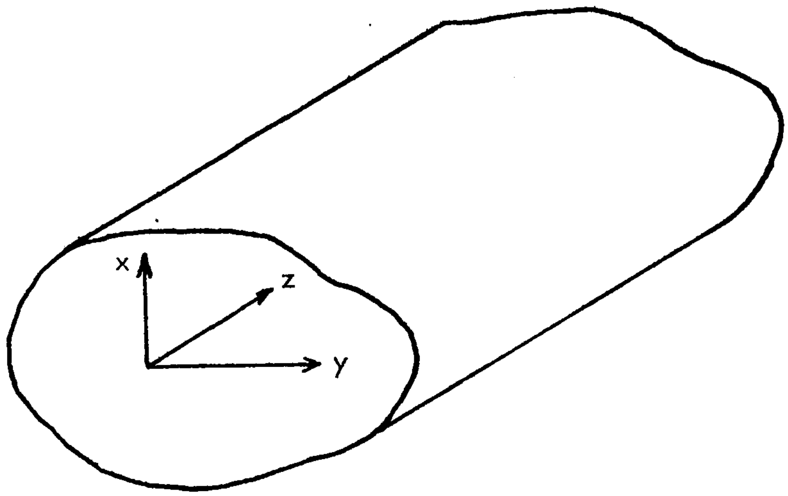
Figure Captions

Fig. 1 (a) A waveguide with arbitrary cross section and the relevant coordinate system.

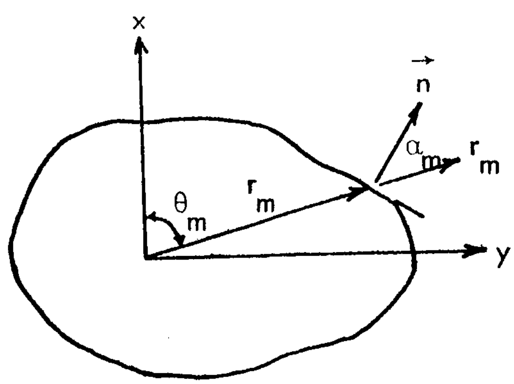
(b) The angle α_m at point (r_m, θ_m) on the cross-sectional contour.

Fig. 2 The square guide with the six chosen points.

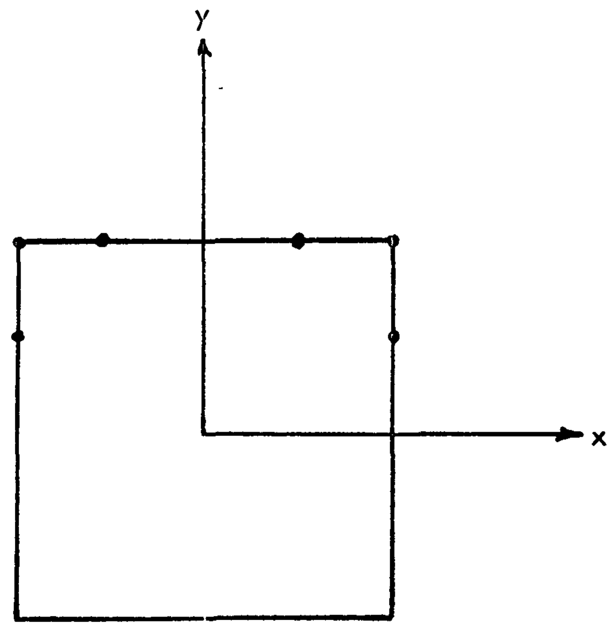
Fig. 3 The attenuation constants of the square waveguide.



x
z
y



x
$\vec{n}$
α_m
r_m
θ_m
r_m
y


y
x

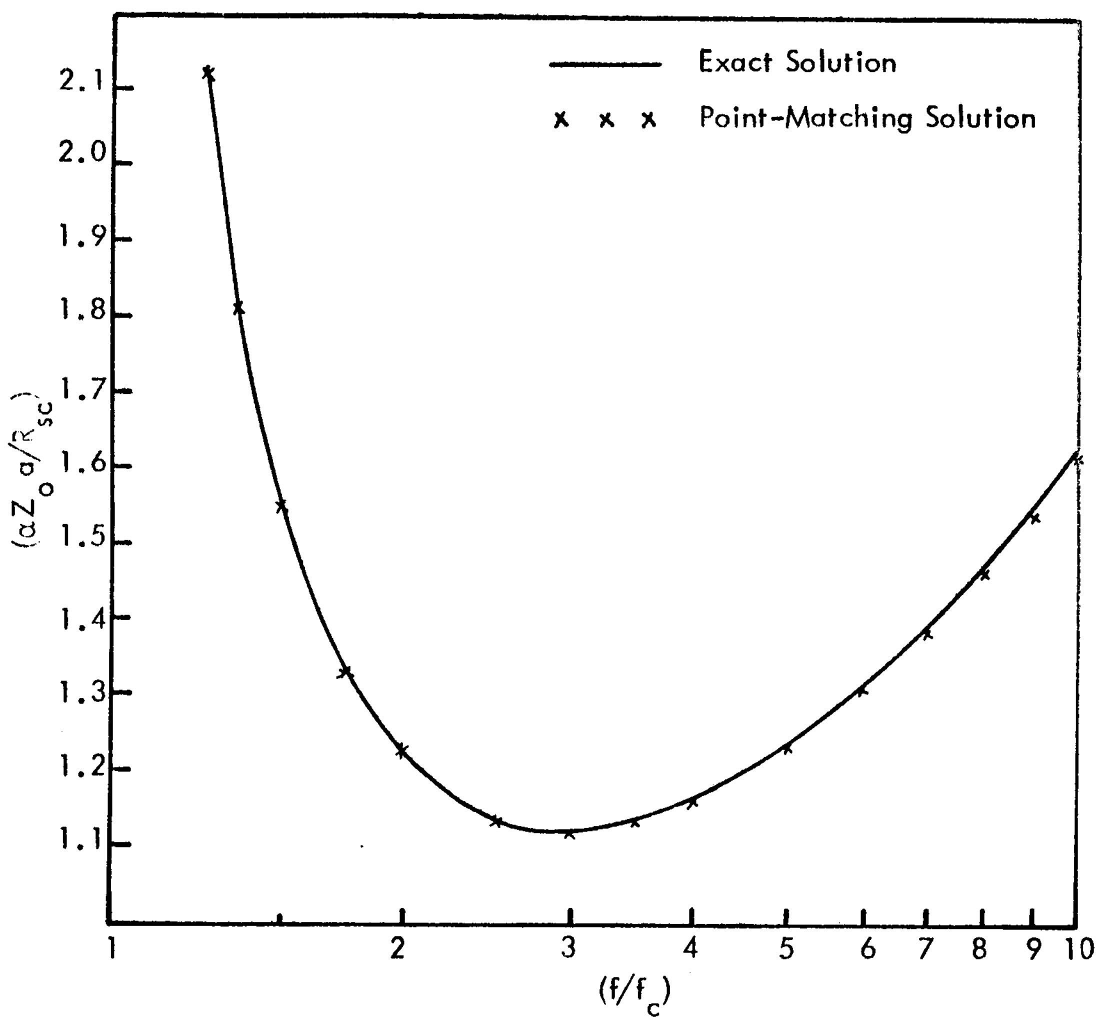

Exact Solution
x x x Point-Matching Solution
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
$(\alpha Z_o a/R_{sc})$
1
2
3
4
5
6
7
8
9
10
(f/f_c)

Table 1. The field distribution (H_z) computed by the point-matching method and compared with the exact solution.

x/a \ y/a	0	0.5	1	Exact
0.1	0.078217	0.078444	0.078194	0.078217
0.2	0.154508	0.154961	0.154470	0.154508
0.3	0.226995	0.227672	0.226955	0.226995
0.4	0.293893	0.294789	0.293867	0.293892
0.5	0.353554	0.354664	0.353562	0.353553
0.6	0.404509	0.405825	0.404567	0.404508
0.7	0.445506	0.447013	0.445624	0.445503
0.8	0.475536	0.477215	0.475711	0.475528
0.9	0.493860	0.495689	0.494075	0.493844
1.0	0.500034	0.501979	0.500246	0.500000